1.0

1.1

1.25 1.4 1.6

2.8 2.5

3.2 2.2

3.6

4.0 2.0

1.8

4.5

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MULTIDIMENSIONAL INFINITELY DIVISIBLE VARIABLES
AND PROCESSES. PART I. STABLE CASE

By

Raoul LePage

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DEPARTMENT OF STATSITICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

MULTIDIMENSIONAL INFINITELY DIVISIBLE VARIABLES
AND PROCESSES. PART I: STABLE CASE

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1. Introduction.

Series decompositions, involving the arrival times of a Poisson
process, have been given by Ferguson and Klass [1] for the non-Gaussian
component of an arbitrary (real-valued) independent increments random
function on the unit interval. LePage and Woodroofe and Zinn [4] have
rediscovered a variant of this decomposition in connection with their
study, via order statistics, of the limit distribution for self-normalized
sums (e.g. Students'-t), when sampling from a distribution in the domain
of attraction of an arbitrary stable law of index $\alpha < 2$.

The present paper obtains a characterization of stable laws on
spaces of dimension greater than one. This characterization is formally
like that of Ferguson-Klass for dimension one, but with i.i.d. vector
multipliers on the Poisson terms. The law of these coefficients may be
chosen proportional to the Lévy measure, although this is not necessary.
These results take a particularly elegant form in the case of symmetric
stable laws, where something of a calculus is developed showing:
(i) which Lévy measure associates with vector coefficients other than
the aforementioned ones, (ii) what happens when independent stables are
linearly combined as in a weighted sum, (iii) how to construct an arbi-
trary multidimensional independent-increments symmetric stable set

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function, and (iv) how to construct an arbitrary harmonizable stationary symmetric stable random function having multidimensional domain and/or range. Symmetric stable laws are shown to be mixtures of Gaussian laws.

Partly because of the self-contained character of Kuelbs' paper [3], in which the characterization of the log-characteristic function of a stable law is extended to real separable Hilbert space, the Hilbert space level of generality has been chosen for this paper. Later extensions of Kuelbs' result to Banach and more general spaces support a corresponding generalization of these results. In addition to Kuelbs' result we need a method employed by Ferguson and Klass to transform certain dependent series into eventually identical independent ones. We also require standard results giving conditions under which an independent series in Hilbert space converges almost surely (e.g. [2], Theorem 5.3). The rest of the paper is basically self-contained and affords a surprisingly accessible and clear view of $\alpha < 2$ stable laws, and random functions, based on elementary series constructions.

Part II of this paper will generalize these results to the infinitely divisible case.

2. Notation.

The following symbols and conventions will be in force throughout this paper.

(2.1)

	$\sim$	"is asymptotic with"
	$\overset{\Delta}{=}$	"equals by definition"
	$\overset{D}{=}$	"has the same distribution as"
	$\Rightarrow$	"converges in distribution to"
	α	$0 < \alpha < 2$, an index of stability
	$\{\Gamma_j, j \geq 1\}$	arrival times of a Poisson process with unit rate
	H	a real separable Hilbert space

The material of the next section is drawn from [4].

3. Motivation.

Limit theorems are not the subject of this paper. However, we should not proceed without benefit of the following example, which exposes some connections between $\alpha < 2$ stable r.v. and the Poisson process.

Let $\{\epsilon_j, j \geq 1\}$ be independent of the sequence Γ and i.i.d. with $P(\epsilon_1 = 1) = P(\epsilon_1 = -1) = \frac{1}{2}$, and define $G(x) \overset{\Delta}{=} x^{-\alpha}$, $\forall x \geq 1$. Think of $G(x) = P(|X| \geq x)$ $\forall x \geq 1$, where X is a r.v. symmetrically distributed about zero. We will construct particular r.v. $X_1,\ldots,X_n$, i.i.d. as X, whose normalized sums converge in distribution to the symmetric stable law of index α. To do this, use the arrival times of a Poisson process to generate uniform order statistics, apply G^{-1} to these, multiply by the signs ϵ, and permute. As constructed, the normalized sums will actually converge almost surely to $\Sigma_1^\infty \epsilon_j \Gamma_j^{-1/\alpha}$ (see (3.1) below), a series possessing the symmetric stable law of index α. In fact, a direct proof of the stability of $\Sigma_1^\infty \epsilon_j \Gamma_j^{-1/\alpha}$ follows easily from the observation that the arrival times of several (say $K > 1$) independent unit rate Poisson processes (run simultaneously) constitute K^{-1} times the arrival times of a unit rate Poisson process. This argument works just as well for ϵ replaced by any vector sequence (provided the series converges) and suggests the multivariate extensions of sections 4 and 5.

For each $n > 1$ let $U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(n)}$ denote the order statistics of i.i.d. random variables $U_1,\ldots,U_n$ which are

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uniformly distributed on $[0,1]$. Then for each fixed $n > 1$, letting $X_j \triangleq \epsilon_j G^{-1}(U_j)$, $\forall j \geq 1$,

$$(3.1) \qquad n^{-1/\alpha} \Sigma_1^n \epsilon_j X_j \overset{D}{=} n^{-1/\alpha} \Sigma_1^n \epsilon_j G^{-1}(U_{(j)})$$

$$= n^{-1/\alpha} \Sigma_1^n \epsilon_j U_{(j)}^{-1/\alpha}$$

$$\overset{D}{=} n^{-1/\alpha} \Sigma_1^n \epsilon_j (\Gamma_j/\Gamma_{n+1})^{-1/\alpha}$$

$$= \underbrace{\Sigma_1^n \epsilon_j \Gamma_j^{-1/\alpha}} \underbrace{(\Gamma_{n+1}/n)^{1/\alpha}}$$

$$\downarrow_{a.s.} {}^{1.} \qquad \downarrow_{a.s.} (SLLN)$$

$$\Sigma_1^\infty \epsilon_j \Gamma_j^{-1/\alpha} \qquad 1$$

It is convenient to refer to $\epsilon_j \Gamma_j$, $j \geq 1$ as the residual order statistics, keeping in mind that the ordering is on decreasing absolute values.

The same example suggests an invariance principle (proved in [4]) for self-normed sums such as $\Sigma_1^n X_j / \sqrt{\Sigma_1^n X_n^2}$ which, regardless of α, converge in distribution to a limit law depending only on the stable attracting X_1. For the r.v. constructed above, $\forall n \geq 1$,

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1. Use $\Gamma_j \sim j$ and $\Sigma_1^\infty j^{-2/\alpha} < \infty$ a.s., and apply the 3-series theorem conditional on the sequence Γ.

$$(3.2) \qquad \frac{\Sigma_1^n X_j}{\sqrt{\Sigma_1^n X_j^2}} \stackrel{D}{=} \frac{\Sigma_1^n \epsilon_j \Gamma_j^{-1/\alpha}}{\sqrt{\Sigma_1^n \Gamma_j^{-2/\alpha}}} \stackrel{a.s.}{\rightarrow} \frac{\Sigma_1^\infty \epsilon_j \Gamma_j^{-1/\alpha}}{\sqrt{\Sigma_1^\infty \Gamma_j^{-2/\alpha}}}$$

That is, the limit law of the t-statistic[2] is that of the t-statistic calculated on the residual order statistics (see also [6]).

Even the construction of stable independent-increments processes can be motivated by means of the same example. We restrict our attention to the homogeneous increments case. Let $[T_j, j \geq 1] \stackrel{D}{=} [U_j, j \geq 1]$, and suppose the sequences T, ϵ, Γ are mutually independent. The partial sum processes $\Sigma_1^{[nt]} X_j$, $0 \leq t \leq 1$, $\forall n \geq 1$ can be effected by independent selections of $X_1, \ldots, X_n$ into subsets of sizes $[nt]$ using multiplication by indicators:

$$(3.3) \qquad I_1^{(n)}(t) \stackrel{\Delta}{=} I(T_1 \leq \frac{[nt]}{n})$$

$$I_j^{(n)}(t) \stackrel{\Delta}{=} I(T_j \leq \frac{[nt] - \Sigma_{i<j} I_i^{(n)}(t)}{n+1-j}) , \quad \forall \ 1 \leq j \leq n .$$

Then for each $n \geq 1$,

$$(3.4) \qquad \{ n^{-1/\alpha} \Sigma_1^{[nt]} X_j, \ t\epsilon[0,1] \} \stackrel{D}{=} \{ \Sigma_1^n I_j^{(n)}(t) \ \epsilon_j \Gamma_j^{-1/\alpha} (\Gamma_{n+1}/n)^{1/\alpha}, \ t\epsilon[0,1] \}$$

$$\downarrow \ a.s. \ in \ D[0,1]$$

$$\{ \Sigma_1^\infty I(T_j \leq t) \epsilon_j \Gamma_j^{-1/\alpha}, \ t\epsilon[0,1] \} .$$

Details of this argument are unpublished.

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2. The square of this t-statistic is simply related to the square of Students'-t, and both have the same limit law.

4. Stable Laws on H.

Suppose $\{X_j, j \geq 1\}$ are i.i.d. random vectors in a real separable Hilbert space H and that the sequences X, Γ are independent. For each $n \geq 1$ denote by K_n the number of arrival times Γ in the interval $[0, a_n]$, $a_n \overset{\Delta}{=} \Sigma_1^n j^{-1}$. This choice of a_n is from [1]. Its advantages will be apparent in what follows.

Remark. Sums of the kind $\Sigma_1^{K_n}$ are for each $n \geq 1$ defined to zero on the event $K_n = 0$. Use $(\ ,\ )$, $\|\ \|$, to denote H inner product and norm.

For each $n \geq 1$, $x \in H$, $c > 0$, (see also [1], pg. 1639),

$$(4.1) \qquad Ee^{i(x, c\Sigma_1^{K_n} X_j \Gamma_j^{-1/\alpha})} = EE^{K_n} e^{ic\Sigma_1^{K_n}(x, X_j)\Gamma_j^{-1/\alpha}}$$

$$= E(a_n^{-1} \int_0^{a_n} Ee^{ic(x, X_1)t^{-1/\alpha}} dt)^{K_n}$$

$$= e^{E \int_0^{a_n}(e^{ic(x, X_1)t^{-1/\alpha}} - 1)dt}$$

$$= e^{Ec^\alpha \alpha \int_{ca_n^{-1/\alpha}}^{\infty} (e^{i(x, X_1)r} - 1) \frac{dr}{r^{1+\alpha}}} .$$

Kuelbs ([3], lemma 2.2) has proved that the log-characteristic function of a (non-Gaussian) stable probability measure on H is necessarily of the following form, for a unique $\beta \in H$ and finite Borel measure σ on $S \overset{\Delta}{=} \{x \in H \mid \|x\| = 1\}$,

(4.2) $i(x,\beta) + \int_S \int_0^\infty (e^{i(x,s)r} - 1 - \frac{i(x,s)r}{1+r^2}) \frac{dr}{r^{1+\alpha}} \sigma(ds)$, $\forall x \in H$.

It is convenient to refer to:

(4.3)   expression (4.2) with $\beta = 0$.

Define $\delta \overset{\Delta}{=} \sigma(S)$, $\mu \overset{\Delta}{=} \sigma/\delta$, and

$$\beta_n \overset{\Delta}{=} \int_S \int_0^\infty \alpha^{-1/\alpha} \delta^{-1/\alpha} a_n^{-1/\alpha} \frac{sr}{1+r^2} \frac{dr}{r^{1+\alpha}} \sigma(ds), \forall n \geq 1 \ .$$

Lemma 4.4   If (4.3) is the characteristic function of a stable law on

H and if the sequence $\{X_j, j \geq 1\}$ is i.i.d. μ and independent of the

sequence Γ , define

$$X^{(n)} \overset{\Delta}{=} \alpha^{-1/\alpha} \delta^{1/\alpha} \sum_1^{K_n} X_j \Gamma_j^{-1/\alpha} - \beta_n, \ \forall n \geq 1 \ .$$

Then $\forall x \in H$, $E \exp i(x, X_n)$ converges to (4.3).

Remark. This result is not altogether satisfactory since convergence of

the series $\{X^{(n)}, \ n \geq 1\}$ is through stochastic times $\{K_n, \ n \geq 1\}$,

and is not yet a.s. in H.   These defects are remedied in Theorem 4.8

below.

Proof. For each $x \in H$, $n \geq 1$, by (4.1),

$$(4.5) \quad \log_e Ee^{i(x,X^{(n)})} = -i(x,\beta_n) + E\delta \int_{\alpha^{-1/\alpha}\delta^{1/\alpha}a_n^{-1/\alpha}}^{\infty} (e^{i(x,X_1)r} - 1) \frac{dr}{r^{1+\alpha}}$$

$$= \int_s \int_{\alpha^{-1/\alpha}\delta^{1/\alpha}a_n^{-1/\alpha}}^{\infty} (e^{i(x,s)r} - 1 - \frac{i(x,s)r}{1+r^2}) \frac{dr}{r^{1+\alpha}} \sigma(ds)$$

$$\to (4.3) \qquad \square$$

From ([1], lemma 2), we conclude that

$$\left\{ \sum_{K_n}^{K_{n+1}} X_j \Gamma_j^{-1/\alpha}, \ n \geq 1 \right\}$$

are independent. Furthermore, using an argument drawn from [4], $\forall n > 1$,

$$(4.6) \qquad E \left\| \sum_{K_n}^{K_{n+1}} X_j \Gamma_j^{-1/\alpha} \right\|^2 \leq a_n^{-2/\alpha} E(K_{n+1} - K_n)^2 \sim < \frac{2}{n(\log_e n)^{2/\alpha}} \,,$$

which is summable in n. Therefore,

$$\sum_1^{K_n} X_j \Gamma_j^{-1/\alpha} - E \sum_1^{K_n} X_j (\Gamma_j^{-1/\alpha} \wedge 1)$$

converges in probability in H. A short calculation gives

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$$(4.7) \quad A_n(\alpha,\delta) \triangleq E \; \alpha^{-1/\alpha} \delta^{1/\alpha} \sum_1^{K_n} (\Gamma_j^{-1/\alpha} \wedge 1) - \delta \int_{\alpha^{-1/\alpha}\delta^{1/\alpha}a_n^{-1/\alpha}}^{\infty} \frac{r}{1+r^2} \frac{dr}{r^{1+\alpha}}$$

$$\overset{\underline{3./}}{=} \alpha^{-1/\alpha}\delta^{1/\alpha} \int_0^{a_n} \left((t^{-1/\alpha} \wedge 1) - \frac{t^{-1/\alpha}}{1+\alpha^{-2/\alpha}\delta^{2/\alpha} t^{-2/\alpha}}\right) dt$$

$$\to \text{finite limit} \; \triangleq A(\alpha,\delta) \; \text{as} \; n \to \infty \; .$$

Therefore $X^{(n)}$ converges in probability.

Theorem 4.8. If (4.3) is the log characteristic function of a stable law on H then the series

$$\alpha^{-1/\alpha}\delta^{1/\alpha} \sum_1^n \{X_j \Gamma_j^{-1/\alpha} - (EX_1) \int_{j-1}^{j} \frac{t^{-1/\alpha}}{1+\alpha^{-2/\alpha}\delta^{2/\alpha} t^{-2/\alpha}}) \; dt\}$$

converges a.s. in H to a random vector with log characteristic function (4.3).

Remark. Centering is not needed for the case $\alpha < 1$, nor is it needed for the symmetric case which will appear in [4].

Proof. Since $X^{(n)}$ converges in probability and $X^{(n)}$ is an independent series, we conclude by ([2], Theorem 5.3(6)) that $X^{(n)}$ converges a.s. in H. Recall that with probability one $\exists$ finite M such that $(n \geq M) \Rightarrow (\exists$ smallest $N(n)$ with $n = K_{N(n)})$. Then $\forall n \geq m$,

---

3. The first term equals the first term above. For the second term use $t = \alpha^{-1}\delta r^{-\alpha}$

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$$(4.9) \qquad X^{(N(n))} = \alpha^{-1/\alpha} \delta^{1/\alpha} \{\Sigma_1^n X_j \Gamma_j^{-1/\alpha} - (EX_1) \int_0^{a_{N(n)}} (t^{-1/\alpha} \wedge 1) dt\} +$$

$$(EX_1) A_n(\alpha, \delta) + o(1) \ .$$

Since for $n \geq M$, $\quad (n = K_{N(n)}) \Rightarrow (a_{N(n)-1} < \Gamma_n \leq a_{N(n)})$,

$$(4.10) \qquad \left| \int_{\Gamma_n}^{a_{N(n)}} (t^{-1/\alpha} \wedge 1) dt \right| \leq (N(n))^{-1} \xrightarrow{\text{a.s.}} 0 \ .$$

By the law of the iterated logarithm, a.s. eventually as $n \to \infty$

$$(4.11) \qquad \left| \int_n^{\Gamma_n} (t^{-1/\alpha} \wedge 1) dt \right| = \int_n^{\Gamma_n} t^{-1/\alpha} dt \leq |\Gamma_n - n| (n \wedge \Gamma_n)^{-1/\alpha}$$

$$\leq 2 \sqrt{n \log \log n} \ (n + o(n))^{-1/\alpha} \to 0 \ .$$

Therefore,

$$(4.12) \qquad X^{(N(n))} = \alpha^{-1/\alpha} \delta^{1/\alpha} \{\Sigma_1^n X_j \Gamma_j^{-1/\alpha} - (EX_1) \int_0^n (t^{-1/\alpha} \wedge 1) dt\} -$$

$$A(\alpha, \delta)(EX_1) + o(1) \ ,$$

which converges a.s. in H to a random vector with log characteristic function (4.3). $\square$

<u>Remark</u>. The centerings used above also have an interpretation involving $(EX_1) \ E(\Gamma_j^{-1/\alpha} \wedge 1)$, which will not be given here.

5.   Symmetric Case, Multiple Representations.

In this section we do not assume that X_1 is distributed according to the measure μ, or even restrict its distribution to S. Suppose X, ϵ, Γ are mutually independent sequences, with ϵ, Γ as in section 3, and $\{X_j, j \geq 1\}$ i.i.d. in H with $E\|X_1\|^\alpha < \infty$.

Remark. Series of form $\Sigma_1^n \epsilon_j X_j \Gamma_j^{-1/\alpha}$ will be termed symmetric.

Lemma 5.1. The symmetric series $\Sigma_1^n \epsilon_j X_j \Gamma_j^{-1/\alpha}$ converges a.s. in H and the log characteristic function of its limit is $E|(x,X_1)|^\alpha B(\alpha)$, $\forall x \in H$, where $B(\alpha) \overset{\Delta}{=} \int_0^\infty \alpha(\cos(r)-1)\,\dfrac{dr}{r^{1+\alpha}}$.

Proof. The arguments needed are similar to those of theorem 4.8, but easier in the symmetric case. For each $x \in H$,

$$(5.2) \quad \log_e E e^{i(x,\Sigma_1^n \epsilon_j X_j \Gamma_j^{-1/\alpha})} \overset{(4.1)}{=} \alpha E \int_{a_n^{-1/\alpha}}^\infty (e^{i\epsilon_1|(x,X_1)|r}-1)\,\frac{dr}{r^{1+\alpha}}$$

$$= \alpha E \int_0^\infty (e^{i\epsilon_1 r}-1)I(|(x,X_1)| \leq ra_n^{1/\alpha})|(x,X_1)|^\alpha \,\frac{dr}{r^{1+\alpha}}$$

$$= \alpha E \int_0^\infty (\cos(r)-1)I(|(x,X_1)| \leq ra_n^{1/\alpha})|(x,X_1)|^\alpha \,\frac{dr}{r^{1+\alpha}}$$

$$\rightarrow \alpha \int_0^\infty (\cos(r)-1)\,\frac{dr}{r^{1+\alpha}}\; E|(x,X_1)|^\alpha \quad .$$

From ([3], corollary 2.1), the limit in (5.2) is the characteristic

function of a (symmetric) stable law on H. Since the sums

$\Sigma_{K_n}^{K_{n+1}} \epsilon_j X_j \Gamma_j^{-1/\alpha}$ are independent and symmetric, we have from ([2],

theorem 5.3(1)) that $\Sigma_1^{K_n} \epsilon_j X_j \Gamma_j^{-1/\alpha}$ converges a.s. in H. Since

eventually a.s. $K_{n+1} \leq K_n + 1$ as $n \to \infty$, we conclude $\Sigma_1^n \epsilon_j X_j \Gamma_j^{-1/\alpha}$

converges a.s. in H. $\square$

Several series may represent the same stable law.

<u>Theorem 5.2</u>. If $E\|X_1\|^\alpha < \infty$ then for every $x \in H$,

$E|(x,X_1)|^\alpha = E\|X_1\|^\alpha E|(x,X_1^*)|^\alpha$ where X_1^* is distributed on S

according to the measure:

$$P(X_1^* \in A) = E I_{\left(\frac{X_1}{\|X_1\|} \in A\right)} \frac{\|X_1\|^\alpha}{E\|X_1\|^\alpha} \quad .$$

<u>Proof</u>. For every $x \in H$,

$$E|(x,X_1)|^\alpha = E|(x, \frac{X_1}{\|X_1\|})|^\alpha \|X_1\|^\alpha$$

$$= E\|X_1\|^\alpha E|(x,X_1^*)|^\alpha . \quad \square$$

As an example of the above, every symmetric stable law on H has
a construction $\Sigma_1^\infty Z_j X_j \Gamma_j^{-1/\alpha}/E(|Z_1|^\alpha)^{1/\alpha}$ in terms of an independent
sequence Z of i.i.d. standard normal r.v.. Conditional on the sequences
Γ and X the symmetric stable is Gaussian. That is, symmetric stable
laws are particular mixtures of Gaussian laws with zero means and differing
covariance kernels. The latter will not in general differ only by scale,
though this is necessarily true for $H = R_1$.

6. Symmetric Stable Random Set Functions.

Let $\{\tau_j, j \geq 1\}$ be i.i.d. taking values in a measurable space with measurable sets generically denoted A. Suppose τ, ϵ, X, Γ are mutually independent, where the latter three sequences are as in section 5. Define

$$(6.1) \qquad X(A) \overset{\Delta}{=} \Sigma_1^\infty I(\tau_j \in A) X_j \epsilon_j \Gamma_j^{-1/\alpha} \ , \ \forall A \ .$$

Theorem 6.2. The series (6.1) is a.s. convergent for each A, is jointly symmetric stable for finitely many A at a time, and $X(A_1), \ldots, X(A_n)$ are for each $n \geq 1$ mutually independent if $A_1, \ldots, A_n$ are mutually disjoint.

Proof. For each $n \geq 1$, $x \in H$, real numbers $r_1, \ldots, r_n$, and measurable sets $A_1, \ldots, A_n$,

$$(6.3) \quad \log_e E \, e^{i(x, \Sigma_1^n r_k X(A_k))} = \log_e E \, e^{i \Sigma_{j=1}^\infty \{\Sigma_{k=1}^n r_k I(\tau_j \in A_k)\}(x, X_j) \epsilon_j \Gamma_j^{-1/\alpha}}$$

$$\overset{(5.1)}{=} B(\alpha) E |(x, X_1)|^\alpha E |\Sigma_{k=1}^n r_k I(\tau_1 \in A_k)|^\alpha \ .$$

If $A_1, \ldots, A_n$ are mutually disjoint then

$$(6.4) \qquad E |\Sigma_{k=1}^n r_k I(\tau_1 \in A_k)|^\alpha = \Sigma_{k=1}^n |r_k|^\alpha P(\tau_1 \in A_k) \ . \quad \square$$

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Remark. The simplicity of this construction is interesting, as is the way in which α-dependence, dimensional structure, and functional dependence are identified with mutually independent coefficient sequences $\Gamma^{-1/\alpha}$, X, I.

Remark. Schilder [8] and Kuelbs [3] have explored a representation of multidimensional symmetric stable r.v. by means of a stochastic integral with respect to a one-dimensional stable independent increments process. Theorem (4.8) and lemma (5.1) sharpen and extend such representations by connecting them with the Ferguson-Klass representation, making explicit the choice of coefficients required to obtain each stable law[4], and establishing H convergence of the indicated series.

Remark. Suppose $K > 1$ and $Y_k = \Sigma_1^\infty \xi_{kj} \Gamma_{kj}^{-1/\alpha}$, $1 \le k \le K$, are independently constructed (as per (5.1)) symmetric stable r.v. taking values in H. Then for an arbitrary choice of real numbers $r_1, \ldots, r_K$ the sum $\Sigma_1^K r_k Y_k$ is representable $\Sigma_1^\infty \xi_j \Gamma_j^{-1/\alpha}$ where $\{\xi_j, j \ge 1\}$ are i.i.d. and $\xi_1 \overset{D}{=}$ an equiprobable random selection from $K^{1/\alpha} r_1 \xi_{11}$, $\ldots, K^{1/\alpha} r_n \xi_{n1}$. This uses the property (discussed in section 3) of K Poisson processes run simultaneously.

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4. Including the non-symmetric stable laws.

7. Harmonizable Stationary Symmetric Stable Random Functions.

Basically, we seek to construct the stable analogues of Gaussian stationary random functions having a harmonic decomposition. The characteristic function of such a Gaussian random function involves

$$E \left| \Sigma_{k=1}^n r_k \; e^{i(\Lambda_1, t_k)_0} \right|^2 ,$$

where t is generic for a point of the domain, and $(\Lambda_1, \;)_0$ is a random linear function on the domain. The stable analogues of these Gaussian random functions have characteristic functions that employ an αth-power in this integral instead of the 2, but are otherwise the same. Define $\forall t$,

$$(7.1) \qquad X(t) \overset{\Delta}{=} \Sigma_1^\infty \cos((\Lambda_j, t)_0 + \Theta_j) X_j \epsilon_j \Gamma_j^{-1/\alpha} ,$$

where Λ are i.i.d., Θ are i.i.d. uniforms on $[-\Pi, \Pi]$, $\{\epsilon, X, \Gamma\}$ are as in section 6, and Λ, Θ, ϵ, X, Γ are mutually independent sequences. The series (7.1) is a.s. convergent in H for each t by lemma 5.1. The random function $X(\;)$ is clearly stationary because Θ are uniform on $[-\Pi, \Pi]$, but this will also be a simple consequence of the form of the characteristic function which we now compute.

$$(7.2) \qquad \log_e E \; e^{i(x, \Sigma_{k=1}^n r_k X(t_k))}$$

$$= \log_e Ee^{i \; \Sigma_{j=1}^\infty \{\Sigma_{k=1}^n r_k \cos((\Lambda_j, t_k)_0 + \Theta_j)\} \epsilon_j (x, X_j) \Gamma_j^{-1/\alpha}}$$

$$\overset{(5.1)}{=} B(\alpha) E |(x, X_1)|^\alpha \; E |\Sigma_{k=1}^n r_k \cos((\Lambda_1, t_k)_0 + \Theta_1)|^\alpha .$$

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The final term in the right side of (7.2) reduces as follows, with

$$z \overset{\Delta}{=} \Sigma_1^n r_k e^{i(\Lambda_1, t_k)_o} \,,$$

(7.3) $\quad E|\Sigma_1^n r_k \cos((\Lambda_1, t_k)_o + \Theta_1)|^\alpha = E2^{-\alpha}|ze^{i\Theta_1} + \bar{z}e^{-i\Theta_1}|^\alpha$

$$= E|z|^\alpha 2^{-\alpha} E^z |1 + z^{-1}\bar{z}e^{-2i\Theta_1}|^\alpha$$

$$= E|z|^\alpha \, 2^{-\alpha} C(\alpha)$$

where

$$C(\alpha) = \int_{-\pi}^{\pi} |1 + e^{i\eta}|^\alpha d\eta = \int_{-\pi}^{\pi} |1 + e^{i\psi - 2i\eta}|^\alpha d\eta$$

for all real ψ. We have therefore proved,

Theorem 7.4. The random function defined by (7.1) converges a.s. in H for each t, and has log characteristic function

$$2^{-\alpha} B(\alpha) C(\alpha) E|\Sigma_{k=1}^n r_k e^{i(\Lambda_1, t_k)_o}|^\alpha$$

for all $n \geq 1$, $r_1, \ldots, r_n$, $t_1, \ldots, t_n$. $\square$

Corollary 7.5. The random function (7.1) is non-ergodic for each $\alpha < 2$.
Proof. By using (7.2) the construction (7.1) remains valid if ε_j are replaced by $z_j/(E|z_1|^\alpha)^{1/\alpha}$, $j \geq 1$, where Z is an independent i.i.d. standard normal sequence. Conditional on the sequences Λ, X, Γ, the

process (7.1) is a.s.   stationary Gaussian with discrete spectrum,
therefore conditionally non-ergodic a.s..

8. Operator Stable and Infinitely Divisible Multidimensional Laws.

Infinitely divisible laws, of which the operator stable laws are a special case with particularly interesting structure, are treated in Part II. In brief, this is what happens: A construction of infinitely divisible random vectors is given by $\Sigma\{X_j H(\Gamma_j, X_j) - \gamma_j\}$, in which the real function H is monotone decreasing and positive for each value X_j, and is determined from the Lévy measure. A construction of full operator stable random vectors in a finite dimensional real vector space R^d is $\Sigma\{A(\Gamma_j^{-1})X_j - \gamma_j\}$, in which the vectors γ are non-stochastic centerings, $\{A(t) = \exp(B \log t), t > 0\}$ is the group of linear transformations figuring in the definition of operator stability (e.g. Sharpe [9]), and the vectors X are i.i.d. from a probability measure (a factor of the Lévy measure) on a set of generators of the subgroups induced by A. The methods of sections 4 and 5 carry over, as will now be indicated. If X is any i.i.d. sequence in R^d, and X is independent of Γ, then $\forall x \in R^d$, $n \geq 1$,

$$(8.1) \quad \log_e E\, e^{i(x, \Sigma_1^{K_n} A(\Gamma_j^{-1})X_j)} = \int_{a_n^{-1}}^{\infty} E(e^{i(x, A(t)X_1)} - 1)\, \frac{dt}{t^2}.$$

As usual, the symmetric case is simplest. If we examine Sharpe's Theorem 5, we discover that the limit of (8.1) is precisely the form taken by the operator stable in this case, provided we choose for the distribution of X_1 the probability measure figuring in Sharpe's representation of the Lévy measure as a mixture, this measure being placed on (Sharpe's notation) generators θ characterized by s $M_\theta\{t^B\theta : t > s\} = s^{-1}$, $\forall s > 0$. Arguing as in section 5, we conclude $\Sigma_1^{\infty} A(\Gamma_j^{-1})X_j$ converges a.s. in R^d and has the log-characteristic function which is the limit of (8.1).

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9. Priority of P. Lévy.

P. Lévy has anticipated the series constructions of one dimensional stable r.v. with $\alpha < 2$. For the case of a positive stable with $\alpha < 1$, up to scale and location, this construction is $\sum\limits_{j=1}^{\infty} \Gamma_j^{-1/\alpha}$, with $\{\Gamma_j, j > 1\}$ being the arrival times of a Poisson process (on R^+) having unit intensity function. Lévy writes the series in the form $\sum\limits_{x} U_x$, where

(9.1) $\qquad \{U_x, x > 0\}$ are independent r.v. and

(9.2) $\qquad P(U_x = x) = \dfrac{\alpha dx}{x^{1+\alpha}} = 1 - P(U_x = 0)$.

Here is my abstract of the key parts of Lévy's (1935) arguments for the above case:

(9.3) $[\int\limits_{x_0}^{\infty} \dfrac{\alpha dx}{x^{1+\alpha}} = x_0^{-\alpha} < \infty] \Rightarrow [\{U_x : U_x \neq 0, x > x_0\}$ is finite for $x_0 > 0]$,

and also,

(9.4) $\quad [\int\limits_0^{x_0} \dfrac{\alpha dx}{x^{\alpha}} = \dfrac{\alpha}{1-\alpha} x_0^{(1-\alpha)} \downarrow 0] \Rightarrow [E \sum\limits_{x < x_0} U_x \downarrow 0]$ (as $x_0 \downarrow 0$) .

Therefore, for arbitrary $c_1, c_2 > 0$ (defining $c_3^{\alpha} = c_1^{\alpha} + c_2^{\alpha}$ and taking independent copies),

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$$(9.5) \qquad c_1 \sum_x U_x^{(1)} + c_2 \sum_x U_x^{(2)} \overset{\text{Dist.}}{=} \sum_x Y_x^{(1)} + \sum Y_x^{(2)}$$

$$\overset{\text{Dist.}}{=} \sum_x Y_x^{(x)}$$

$$\overset{\text{Dist.}}{=} c_3 \sum_x U_x \qquad (\Rightarrow \text{ stable}),$$

where $\{Y_x^{(k)}, x > 0\}$ have respective intensities $c_k^\alpha \, \alpha \, dx/x^{1+\alpha}$, $k = 1, 2, 3$, and are independent for $k = 1, 2$.

The above arguments do yield a proof of the representation if we apply them to the independent sub-sums $\sum \{U_x : x \in [b_{n+1}, \, b_n]\}$, $n \geq 1$, where $b_n^{-\alpha} = \log n$. This is essentially the argument of Ferguson-Klass (1972). The particular choice of b_n, $n \geq 1$ is one which ensures that eventually as $n \to \infty$ each sub-sum contains at most one summand, so it really is (almost) as though one could add independent U_x one at a time toward $x \to 0$. A quite different justification is to interpret $\sum_x U_x$ as a generalized process driven by "white noise" $\{U_x, x > 0\}$.

Lévy's observations are easily overlooked. Ferguson-Klass, Vervaat (1979), LePage-Woodroofe-Zinn (1979) (in manuscript form), rediscover the Lévy construction as byproducts of the following independent pursuits respectively: (F-K) - representing the positive non-Gaussian part of an independent increments random function as the sum of its ordered jumps. (V)-examining a shot-noise associated with the asymptotic behavior of the solution of a stochastic difference equation as time is increased, (L-W-Z)-studying the limit behavior of the normalized order statistics

from a distribution attracted to a stable. Resnick (1976) reconciles the
Ferguson-Klass construction with the Ito representation, meaning by the
latter Ito's generalization of Lévy's stochastic integral construction
by a Poisson random measure.

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Elementary series constructions, involving a Poisson process, are obtained for multidimensional stable variables and random functions. Symmetric stable laws are shown to be mixtures of Gaussian laws.

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